



# Mathematics HL and further mathematics HL formula booklet

For use during the course and in the examinations  
First examinations 2014

Edited in 2015 (version 2)

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## Prior learning

Area of a parallelogram	$A = b \times h$ , where $b$ is the base, $h$ is the height
Area of a triangle	$A = \frac{1}{2}(b \times h)$ , where $b$ is the base, $h$ is the height
Area of a trapezium	$A = \frac{1}{2}(a + b)h$ , where $a$ and $b$ are the parallel sides, $h$ is the height
Area of a circle	$A = \pi r^2$ , where $r$ is the radius
Circumference of a circle	$C = 2\pi r$ , where $r$ is the radius
Volume of a pyramid	$V = \frac{1}{3}(\text{area of base} \times \text{vertical height})$
Volume of a cuboid	$V = l \times w \times h$ , where $l$ is the length, $w$ is the width, $h$ is the height
Volume of a cylinder	$V = \pi r^2 h$ , where $r$ is the radius, $h$ is the height
Area of the curved surface of a cylinder	$A = 2\pi r h$ , where $r$ is the radius, $h$ is the height
Volume of a sphere	$V = \frac{4}{3}\pi r^3$ , where $r$ is the radius
Volume of a cone	$V = \frac{1}{3}\pi r^2 h$ , where $r$ is the radius, $h$ is the height
Distance between two points $(x_1, y_1)$ and $(x_2, y_2)$	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints $(x_1, y_1)$ and $(x_2, y_2)$	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Solutions of a quadratic equation	The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## Topic I: Algebra

1.1	<p>The <math>n</math>th term of an arithmetic sequence</p> <p>The sum of <math>n</math> terms of an arithmetic sequence</p> <p>The <math>n</math>th term of a geometric sequence</p> <p>The sum of <math>n</math> terms of a finite geometric sequence</p> <p>The sum of an infinite geometric sequence</p>	$u_n = u_1 + (n-1)d$ $S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$ $u_n = u_1 r^{n-1}$ $S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$ $S_\infty = \frac{u_1}{1 - r}, \quad  r  < 1$
1.2	Exponents and logarithms	$a^x = b \Leftrightarrow x = \log_a b, \text{ where } a > 0, b > 0, a \neq 1$ $a^x = e^{x \ln a}$ $\log_a a^x = x = a^{\log_a x}$ $\log_b a = \frac{\log_c a}{\log_c b}$
1.3	<p>Combinations</p> <p>Permutations</p> <p>Binomial theorem</p>	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ ${}^n P_r = \frac{n!}{(n-r)!}$ $(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{r} a^{n-r} b^r + \dots + b^n$
1.5	Complex numbers	$z = a + ib = r(\cos \theta + i \sin \theta) = r e^{i\theta} = r \operatorname{cis} \theta$
1.7	De Moivre's theorem	$[r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta) = r^n e^{in\theta} = r^n \operatorname{cis} n\theta$

## Topic 2: Functions and equations

<b>2.5</b>	Axis of symmetry of the graph of a quadratic function	$f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry $x = -\frac{b}{2a}$
<b>2.6</b>	Discriminant	$\Delta = b^2 - 4ac$

## Topic 3: Circular functions and trigonometry

<b>3.1</b>	Length of an arc  Area of a sector	$l = \theta r$ , where $\theta$ is the angle measured in radians, $r$ is the radius  $A = \frac{1}{2}\theta r^2$ , where $\theta$ is the angle measured in radians, $r$ is the radius
<b>3.2</b>	Identities    Pythagorean identities	$\tan \theta = \frac{\sin \theta}{\cos \theta}$  $\sec \theta = \frac{1}{\cos \theta}$  $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$  $\cos^2 \theta + \sin^2 \theta = 1$ $1 + \tan^2 \theta = \sec^2 \theta$ $1 + \cot^2 \theta = \operatorname{csc}^2 \theta$
<b>3.3</b>	Compound angle identities    Double angle identities	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$  $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$  $\sin 2\theta = 2 \sin \theta \cos \theta$ $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$  $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

<b>3.7</b>	Cosine rule	$c^2 = a^2 + b^2 - 2ab \cos C$ ; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
	Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
	Area of a triangle	$A = \frac{1}{2} ab \sin C$

## Topic 4: Vectors

<b>4.1</b>	Magnitude of a vector	$ \mathbf{v}  = \sqrt{v_1^2 + v_2^2 + v_3^2}$ , where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
	Distance between two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
	Coordinates of the midpoint of a line segment with endpoints $(x_1, y_1, z_1)$ , $(x_2, y_2, z_2)$	$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$
<b>4.2</b>	Scalar product	$\mathbf{v} \cdot \mathbf{w} =  \mathbf{v}   \mathbf{w}  \cos \theta$ , where $\theta$ is the angle between $\mathbf{v}$ and $\mathbf{w}$
	Angle between two vectors	$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$ , where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ , $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\cos \theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{ \mathbf{v}   \mathbf{w} }$
<b>4.3</b>	Vector equation of a line	$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$
	Parametric form of the equation of a line	$x = x_0 + \lambda l$ , $y = y_0 + \lambda m$ , $z = z_0 + \lambda n$
	Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$

<b>4.5</b>	<p>Vector product</p> $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} \text{ where } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ <p><math> \mathbf{v} \times \mathbf{w}  =  \mathbf{v}   \mathbf{w}  \sin \theta</math>, where <math>\theta</math> is the angle between <math>\mathbf{v}</math> and <math>\mathbf{w}</math></p> <p>Area of a triangle</p> $A = \frac{1}{2}  \mathbf{v} \times \mathbf{w}  \text{ where } \mathbf{v} \text{ and } \mathbf{w} \text{ form two sides of a triangle}$
<b>4.6</b>	<p>Vector equation of a plane</p> $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$ <p>Equation of a plane (using the normal vector)</p> $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ <p>Cartesian equation of a plane</p> $ax + by + cz = d$

## Topic 5: Statistics and probability

<b>5.1</b>	<p><b>Population parameters</b></p> <p>Let <math>n = \sum_{i=1}^k f_i</math></p> <p>Mean <math>\mu</math></p> $\mu = \frac{\sum_{i=1}^k f_i x_i}{n}$ <p>Variance <math>\sigma^2</math></p> $\sigma^2 = \frac{\sum_{i=1}^k f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^k f_i x_i^2}{n} - \mu^2$ <p>Standard deviation <math>\sigma</math></p> $\sigma = \sqrt{\frac{\sum_{i=1}^k f_i (x_i - \mu)^2}{n}}$
<b>5.2</b>	<p>Probability of an event <math>A</math></p> $P(A) = \frac{n(A)}{n(U)}$ <p>Complementary events</p> $P(A) + P(A') = 1$
<b>5.3</b>	<p>Combined events</p> $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ <p>Mutually exclusive events</p> $P(A \cup B) = P(A) + P(B)$

5.4	<p>Conditional probability</p> <p>Independent events</p> <p>Bayes' theorem</p>	$P(A B) = \frac{P(A \cap B)}{P(B)}$ $P(A \cap B) = P(A)P(B)$ $P(B A) = \frac{P(B)P(A B)}{P(B)P(A B) + P(B')P(A B')}$ $P(B_i A) = \frac{P(B_i)P(A B_i)}{P(B_1)P(A B_1) + P(B_2)P(A B_2) + P(B_3)P(A B_3)}$
5.5	<p>Expected value of a discrete random variable <math>X</math></p> <p>Expected value of a continuous random variable <math>X</math></p> <p>Variance</p> <p>Variance of a discrete random variable <math>X</math></p> <p>Variance of a continuous random variable <math>X</math></p>	$E(X) = \mu = \sum x P(X = x)$ $E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$ $\text{Var}(X) = E(X - \mu)^2 = E(X^2) - [E(X)]^2$ $\text{Var}(X) = \sum (x - \mu)^2 P(X = x) = \sum x^2 P(X = x) - \mu^2$ $\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$
5.6	<p>Binomial distribution</p> <p>Mean</p> <p>Variance</p> <p>Poisson distribution</p> <p>Mean</p> <p>Variance</p>	$X \sim B(n, p) \Rightarrow P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$ $E(X) = np$ $\text{Var}(X) = np(1-p)$ $X \sim \text{Po}(m) \Rightarrow P(X = x) = \frac{m^x e^{-m}}{x!}, \quad x = 0, 1, 2, \dots$ $E(X) = m$ $\text{Var}(X) = m$
5.7	Standardized normal variable	$z = \frac{x - \mu}{\sigma}$



## Topic 6: Calculus

<b>6.1</b>	Derivative of $f(x)$	$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \left( \frac{f(x+h) - f(x)}{h} \right)$
<b>6.2</b>	Derivative of $x^n$	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
	Derivative of $\sin x$	$f(x) = \sin x \Rightarrow f'(x) = \cos x$
	Derivative of $\cos x$	$f(x) = \cos x \Rightarrow f'(x) = -\sin x$
	Derivative of $\tan x$	$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$
	Derivative of $e^x$	$f(x) = e^x \Rightarrow f'(x) = e^x$
	Derivative of $\ln x$	$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$
	Derivative of $\sec x$	$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$
	Derivative of $\csc x$	$f(x) = \csc x \Rightarrow f'(x) = -\csc x \cot x$
	Derivative of $\cot x$	$f(x) = \cot x \Rightarrow f'(x) = -\csc^2 x$
	Derivative of $a^x$	$f(x) = a^x \Rightarrow f'(x) = a^x (\ln a)$
	Derivative of $\log_a x$	$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a}$
	Derivative of $\arcsin x$	$f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$
	Derivative of $\arccos x$	$f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1-x^2}}$
	Derivative of $\arctan x$	$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$
	Chain rule	$y = g(u), \text{ where } u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
	Product rule	$y = uv \Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$
	Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

6.4	Standard integrals	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ $\int \frac{1}{x} dx = \ln x  + C$ $\int \sin x dx = -\cos x + C$ $\int \cos x dx = \sin x + C$ $\int e^x dx = e^x + C$ $\int a^x dx = \frac{1}{\ln a} a^x + C$ $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C, \quad  x  < a$
6.5	Area under a curve  Volume of revolution (rotation)	$A = \int_a^b y dx \text{ or } A = \int_a^b x dy$ $V = \int_a^b \pi y^2 dx \text{ or } V = \int_a^b \pi x^2 dy$
6.7	Integration by parts	$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \text{ or } \int u dv = uv - \int v du$

## Topic 7: Statistics and probability

### Further mathematics HL topic 3

<b>7.1</b> <b>(3.1)</b>	Probability generating function for a discrete random variable $X$	$G(t) = E(t^x) = \sum_x P(X = x)t^x$
		$E(X) = G'(1)$
		$\text{Var}(X) = G''(1) + G'(1) - (G'(1))^2$
<b>7.2</b> <b>(3.2)</b>	Linear combinations of two independent random variables $X_1, X_2$	$E(a_1X_1 \pm a_2X_2) = a_1E(X_1) \pm a_2E(X_2)$
		$\text{Var}(a_1X_1 \pm a_2X_2) = a_1^2 \text{Var}(X_1) + a_2^2 \text{Var}(X_2)$
<b>7.3</b> <b>(3.3)</b>	Sample statistics	
Mean $\bar{x}$		$\bar{x} = \frac{\sum_{i=1}^k f_i x_i}{n}$
Variance $s_n^2$		$s_n^2 = \frac{\sum_{i=1}^k f_i (x_i - \bar{x})^2}{n} = \frac{\sum_{i=1}^k f_i x_i^2}{n} - \bar{x}^2$
Standard deviation $s_n$		$s_n = \sqrt{\frac{\sum_{i=1}^k f_i (x_i - \bar{x})^2}{n}}$
Unbiased estimate of population variance $s_{n-1}^2$		$s_{n-1}^2 = \frac{n}{n-1} s_n^2 = \frac{\sum_{i=1}^k f_i (x_i - \bar{x})^2}{n-1} = \frac{\sum_{i=1}^k f_i x_i^2}{n-1} - \frac{n}{n-1} \bar{x}^2$
<b>7.5</b> <b>(3.5)</b>	Confidence intervals	
Mean, with known variance		$\bar{x} \pm z \times \frac{\sigma}{\sqrt{n}}$
Mean, with unknown variance		$\bar{x} \pm t \times \frac{s_{n-1}}{\sqrt{n}}$
<b>7.6</b> <b>(3.6)</b>	Test statistics	
Mean, with known variance		$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$

	Mean, with unknown variance	$t = \frac{\bar{x} - \mu}{s_{n-1} / \sqrt{n}}$
<b>7.7</b> <b>(3.7)</b>	Sample product moment correlation coefficient	$r = \frac{\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}}{\sqrt{\left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right)\left(\sum_{i=1}^n y_i^2 - n\bar{y}^2\right)}}$
	Test statistic for $H_0: \rho = 0$	$t = r\sqrt{\frac{n-2}{1-r^2}}$
	Equation of regression line of $x$ on $y$	$x - \bar{x} = \left( \frac{\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^n y_i^2 - n\bar{y}^2} \right) (y - \bar{y})$
	Equation of regression line of $y$ on $x$	$y - \bar{y} = \left( \frac{\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^n x_i^2 - n\bar{x}^2} \right) (x - \bar{x})$

## Topic 8: Sets, relations and groups

### Further mathematics HL topic 4

<b>8.1</b> <b>(4.1)</b>	De Morgan's laws	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$
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## Topic 9: Calculus

### Further mathematics HL topic 5

<b>9.5</b> <b>(5.5)</b>	Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n); \quad x_{n+1} = x_n + h, \text{ where } h \text{ is a constant (step length)}$
	Integrating factor for $y' + P(x)y = Q(x)$	$e^{\int P(x) dx}$

<b>9.6</b> <b>(5.6)</b>	Maclaurin series	$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$
	Taylor series	$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$
	Taylor approximations (with error term $R_n(x)$ )	$f(x) = f(a) + (x-a)f'(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x)$
	Lagrange form	$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$ , where $c$ lies between $a$ and $x$
	Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \dots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

## Topic 10: Discrete mathematics

### Further mathematics HL topic 6

<b>10.7</b> <b>(6.7)</b>	Euler's formula for connected planar graphs	$v - e + f = 2$ , where $v$ is the number of vertices, $e$ is the number of edges, $f$ is the number of faces
	Planar, simple, connected graphs	$e \leq 3v - 6$ for $v \geq 3$ $e \leq 2v - 4$ if the graph has no triangles

## Formulae for distributions

Topics 5.6, 5.7, 7.1, further mathematics HL topic 3.1

### Discrete distributions

Distribution	Notation	Probability mass function	Mean	Variance
Geometric	$X \sim \text{Geo}(p)$	$pq^{x-1}$ for $x = 1, 2, \dots$	$\frac{1}{p}$	$\frac{q}{p^2}$
Negative binomial	$X \sim \text{NB}(r, p)$	$\binom{x-1}{r-1} p^r q^{x-r}$ for $x = r, r+1, \dots$	$\frac{r}{p}$	$\frac{rq}{p^2}$

### Continuous distributions

Distribution	Notation	Probability density function	Mean	Variance
Normal	$X \sim \text{N}(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\mu$	$\sigma^2$

## Topic I: Linear algebra

1.2	Determinant of a $2 \times 2$ matrix	$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \mathbf{A} =  \mathbf{A}  = ad - bc$
	Inverse of a $2 \times 2$ matrix	$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$
	Determinant of a $3 \times 3$ matrix	$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \Rightarrow \det \mathbf{A} = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$